UNIVERSITY OF THE STATE OF BAHIA
Department of Exact and Earth Sciences - *Campus* II
Postgraduate Program in Biosystems Modeling and Simulation

ALTERNATIVE FRACTAL INDEXES TO THE *BOX-COUNTING DIMENSION* : INFORMATION DIMENSION AND LACUNARITY

Soraia Bitencourt Carvalho

Alagoinhas - Bahia - Brazil
2024

UNIVERSITY OF THE STATE OF BAHIA
Postgraduate Program in Biosystems Modeling and Simulation

Soraia Bitencourt Carvalho

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Dissertation presented to the Postgraduate Program in Biosystems Modeling and Simulation at the State University of Bahia as part of the requirements for obtaining the title of Master in Biosystems Modeling and Simulation.

Area of knowledge: Interdisciplinary.

Research Line: Modeling and Optimization of Biosystems

Advisor: Dr. Antônio Teófilo Ataíde do Nascimento

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Prof. Dr. Adriana Marmori Lima

Dean

Prof. Dr. Tânia Hetkowski

Vice-Dean of Research and Postgraduate Teaching

C331i Carvalho, Soraia Bitencourt

Alternative Fractal Indices to the *Box-Counting Dimension* : Information and Gap Dimension/ Soraia Bitencourt Carvalho – Alagoinhas, 2024.

56 f.: il

Advisor: Prof. Dr. Antônio Teófilo Ataíde do Nascimento.

Thesis (Master's degree) – University of the State of Bahia, Department of Exact and Earth Sciences - Campus II, Postgraduate Program in Biosystems Modeling and Simulation. Master in Biosystems Modeling and Simulation– Alagoinhas, 2024.

1. Information Dimension 2. Lacunarity 3. Box-Counting Dimension I. Nascimento, Antônio Teófilo Ataíde. II. Brito, Gustavo Reis de. III. University of the State of Bahia – Department of Exact and Earth Sciences – Campus II. IV. Title

FOLHA DE APROVAÇÃO

"ÍNDICES FRACTAIS ALTERNATIVOS À DIMENSÃO BOX-COUNTING: DIMENSÃO DA INFORMAÇÃO E LACUNARIDADE "

SORAIA BITENCOURT CARVALHO

Dissertação apresentada ao Programa de Pós-Graduação em Modelagem e Simulação de Biossistemas – PPGMSB, em 8 de março de 2024, como requisito parcial para obtenção do grau de Mestra em Modelagem e Simulação de Biossistemas pela Universidade do Estado da Bahia, conforme avaliação da Banca Examinadora:

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gov.br ANTONIO TEOFILO ATAIDE DO NASCIMENTO
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Verifique em https://validar.iti.gov.br

Professor Dr. ANTÔNIO TEÓFILO ATAÍDE DO NASCIMENTO
UNEB
Doutorado em Matemática
Universidade Federal da Bahia

Documento assinado digitalmente
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Professor(a) Dr.(a) ELTAMARA SOUZA DA CONCEIÇÃO
UNEB
Doutorado em Entomologia
Universidade Federal de Viçosa

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Professor Dr. PERFILINO EUGÊNIO FERREIRA JUNIOR
UFBA
Doutorado em Matemática
Instituto Nacional de Matemática Pura e Aplicada

THANKS

After two years of much study, obstacles, trials, joys, discoveries, partnerships, disappointments and enchantments, I thank God for this entire research process, because without Him nothing would be worth it, and for having His permission to carry out another dream.

I need to express all my gratitude and love to my parents, who taught me since I was a child to look for the best path to follow, never forgetting honesty and my value as a person. To my family, with lots of love, to my daughters and granddaughter, together we are the overflow of God's presence in our lives. Daniela Karine, my encourager, friend, cousin and of great importance at this moment. Most of all, I thank you for believing in me.

To all my friends, thank you very much for experiencing this moment of research with me. Be it cheering, supporting or even helping me in carrying out this work.

To my advisor, Prof. Dr. Antônio Teófilo Ataíde do Nascimento, thank you very much, for accepting me as your advisor, dedicating hours of study to my professional and personal development, strengthening me when everything seemed difficult, helping me when I felt discouraged, always pushing me to grow as a researcher and person. Thank you for your dedication to our work, often leaving your personal time to help me. And in times of difficulty, always finding a way out to carry out the research.

To PPGMSB and all the teachers who contributed to my academic development.

Summary

This work makes use of the information dimension based on Shannon and Tsallis entropy to analyze the contours of flat objects, citing the lacunarity method, which serves to evaluate the distribution of gaps in a data set. The research aimed to apply the information dimension in detecting boundary disturbances and the lacunarity index in the analysis of empty spaces. To this end, examples of squares were constructed with small boundary disturbances, with the same mass and which had the same *Box-Counting* Dimension , but did not have the same information dimension. This construction was designed considering that entropy is sensitive to the frequency of the image in each *box* . In summary, the results obtained demonstrate that the information dimension represents a valuable tool in the analysis of the detection of boundary disturbances, consequently, in the analysis of the complexity of objects, while the lacunarity index in FracDim corresponded to the expected , its lacunarity curves in this proposal it is dependent on the scale parameter (r).

Keywords: information dimension ; lacunarity; *Box-Counting* dimension ; entropy.

ABSTRACT

This work makes use of the information dimension based on Shannon and Tsallis entropy to analyze the contours of flat objects, citing the lacunarity method, which serves to evaluate the distribution of gaps in a data set. The research aimed to apply the information dimension in detecting boundary disturbances and the lacunarity index in the analysis of empty spaces. To this end, examples of squares were constructed with small boundary disturbances, with the same mass and which had the same Box - Counting Dimension, but they did not have the same information dimension. This construction had designed considering that entropy is sensitive to the frequency of the image in each box. In summary, the results obtained demonstrate that the information dimension represents a valuable tool in the analysis of the detection of boundary disturbances, consequently, in the analysis of the complexity of objects, while the lacunarity index in FracDim corresponded to what was expected, its lacunarity curves in this proposal it is dependent on the scale(r) parameter.

Keywords: information dimension; lacunarity; Box-Counting dimension; entropy.

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List of Acronyms

D_{Box} – *Box-Counting* Dimension

DI – Information Dimension

DS – Information Dimension with Shannon entropy

D_{TS} – Information Dimension with Tsallis entropy

H(x) – Shannon Entropy

H_{TS} – Tsallis Entropy

SUMMARY

INTRODUCTION

Many natural phenomena are irregular and can appear equally complex at different scales of observation. The use of fractals to measure the complexity of a phenomenon, or objects in nature, is becoming more and more widespread. And this is mainly due to the broad verification of self-similarities in nature. It can be understood that a fractal is a complex geometric shape, constructed of smaller copies of itself.

The word 'fractal' was coined by Mandelbrot in his essay, coming from the Latin *fractus* , meaning broken, to describe objects that were too irregular to fit into a traditional geometric configuration (Falconer , 2014).

Mandelbrot (1983) defined fractals as conceptual objects that have a similar structure at all spatial scales; they are self-similar and independent of scale. It can be understood that a fractal is a complex geometric shape, constructed of smaller copies of itself.

Fractals are mathematical objects that cannot be modeled purely using the Euclidean geometry approach, however it is possible to associate a fractal with a number, called the fractal dimension, with the characteristic of having a fractional dimension, which gives information about how its shape (nature topological), occupies its *habitat* (Euclidean space). Furthermore, fractal dimension quantifies how the “size” of a fractal set changes with different scales of observation.

The fractal dimension, with the principle of characterizing complex mathematical shapes known as fractals, is a measure of how much these shapes fill space. However, different shapes can have the same value of the fractal dimension given by *Box-Counting.* Therefore, in this study we seek to verify the existence of an adapted index that can resolve certain ambiguities in fractal dimensions, more specifically, in the *Box-Counting dimension* .

The purpose of this study is to apply the dimension of information to detect boundary disturbances and analyze the lacunarity index through the FracDim application. And the specific objectives are: to show that the information dimension, an adaptation of *Box-Counting* , given by Shannon entropy, is capable of detecting small boundary disturbances that are not detected by *Box- Counting* ; to analyze, for the first time, the Tsallis entropy to enhance the effects of the information dimension and evaluate the lacunarity method in a data set to detect changes in the distribution of gaps.

In experimental proof that the information dimension can be applied to distinguish boundary disturbances from flat objects, it is observed that this technique can be applied to ecology such as detecting pathologies in cells, leaves and even monitoring forests.

As a complement to the fractal dimension, Mandelbrot (1983) introduced the concept of lacunarity (a word derived from the Latin “lacuna”, which means “empty”) to describe and quantify the deviation of fractal objects from their translational invariance, evaluating the size distribution of gaps in object texture.

To this end, the analysis of this study is expanded with the calculation of lacunarity to evaluate whether the method can detect changes in patterns with different pixel distributions in the Fracdim application , *a software* under development . And the higher the lacunarity index, the more homogeneous the object of study and the analysis is obtained from the distribution of empty spaces (gaps) left by the object.

This investigation arises due to the researcher's need to place meanings on mathematics content, linked to knowledge in mathematical modeling and the concern for methods that come closer to mathematics in the technological world. Therefore, this study becomes relevant with the purpose of investigating an index that can resolve certain ambiguities in the fractal dimension, contributing to the development of an application that reads the dimensions, based on the images under study and exposes the graphics in a direct and precise.

This dissertation is organized with the following chapters: Chapter I presents data with information on the areas of study, concepts of fractals, fractal dimension and most common dimensions. Chapter II contains the materials and methods for calculating dimensions, mathematical models and *software* used. Chapter III addresses the gap, defining and presenting the implementation. Chapter IV describes the development of the FracDim application. In Chapter V, the results are cited in graphs and tables with discussion. And finally the conclusion, which presents the results obtained in the research on screen.

CHAPTER I. STATE OF THE ART

The reading, organization of texts on fractal dimension and mathematical modeling, as well as the results with ambiguities obtained from the *Box-Counting dimension* provide *insights* for the analysis of the information dimension and lacunarity indices and these can contribute to a more accurate measurement in ecological studies, so that knowledge of mathematics and computational science is reconstructed, aiming to develop skills in ecology.

1.1 FRACTALS

Different definitions of fractals were developed with the evolution of their theory. Fractal is a neologism that arose from the Latin adjective *fractus* and means “irregular” or “broken”. One of the characteristics of fractal objects is the fractal dimension which can be determined with various mathematical formulas.

Falconer (2014) explains that it is not an exact definition of a fractal, just a list of characteristics that are common to most fractals, although there are those that do not have some of these characteristics.

Thus, one can think of a fractal as a set that generally has a fine structure, that is, detail on tiny scales; it is too irregular to be described by traditional geometric or analytical language; is defined simply, usually recursively; presents some form of self-similarity and has a fractal dimension greater than the topological one (Falconer , 2014).

1.2 FRACTAL DIMENSION

The fractal dimension measures the degree of space filling, that is, it quantifies the size of a fractal set (F), thus providing rich information about its geometric properties.

An original definition for fractal dimension presented by Mandelbrot (1983) is the Hausdorff dimension , however there are others that can be applied to any set, without exact self-similarity.

Statistical self-similarity refers to repetitions related to the overall complexity scale, but not the exact pattern. Specifically, details at a given scale are similar, although not identical, to those seen at coarser or finer scales (Seuront , 2010).

Hausdorff dimension comes from the German mathematician Felix Hausdorff (1868-1942), who published works in the area of topology and introduced the idea of this dimension. Years later, the French mathematician Benoit Mandelbrot (1924-2010) resumed Hausdorff 's studies in his work on fractal geometry.

Hausdorff dimension is the oldest and most important of the fractal dimensions and, although it can be defined for any set, its computational estimation is sometimes very difficult (Falconer , 2014). A disadvantage of the Hausdorff dimension is that, in many cases, it is difficult or even impossible to calculate it, even difficult to make a numerical estimate.

Hausdorff dimension is defined as the upper limit of the logarithm of the number of size cubes needed to cover the structure, divided by the logarithm of the size of the cubes. In other words, the Hausdorff dimension measures the amount of detail present in a structure, the higher the Hausdorff dimension , the more complex the structure.

The great advantage of the Hausdorff dimension is its precise definition, which is essential for understanding the dimension as a measure of space filling.

A common property in fractals is *self-similarity* or scale similarity. This term describes the property that many systems possess, in which the parts, when enlarged, will prove to be identical to the system as a whole (Lorenz, 1996).

However, in the fractal dimension, the object can be characterized, firstly the dimension as the number that informs how dense the set occupies the metric space, where it is located and, secondly, it shows the irregularity of its contour, thus locating it. if mathematical objects that have a fractional dimension.

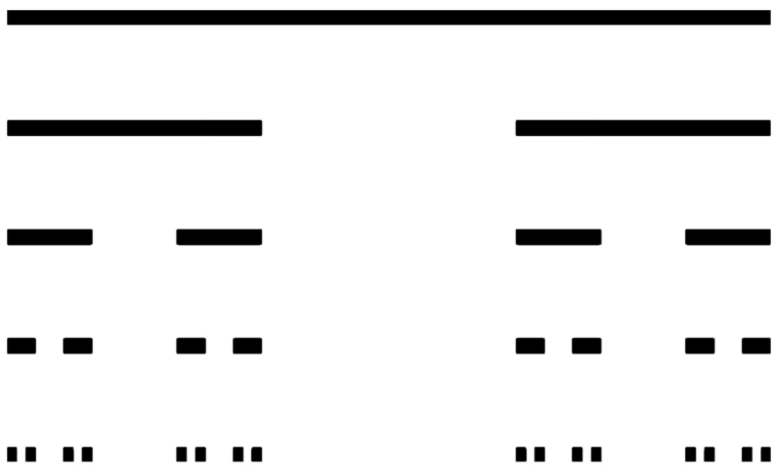
The fractal dimension can take fractional values between 0 and 2 in the plane. And a fractal can be defined as an object, which has a dimension greater than its topological dimension. This measure is applied to characterize self-similar forms . Its usefulness is to provide information about how much a shape fills the plane.

It is possible to associate a fractal with a number, called fractal dimension, which gives information on how its shape (topological nature) occupies its *habitat* (Euclidean space). Intuitively, a point has dimension 0, a straight line has dimension 1, a plane has dimension 2 and a cube has dimension 3.

The Cantor set, the Sierpinski triangle and the Menger sponge , presented in Figures 1, 2 and 3, are found in one-dimensional, two-dimensional and three-dimensional Euclidean space, respectively.

The Cantor Set is an infinite subset of the interval [0,1], obtained by dividing this interval into three equal parts, removing the middle third and repeating this procedure, successively and indefinitely, in the remaining intervals. The points that remain after these infinite successions form the Singer Set.

Figure 1 – Singer Set

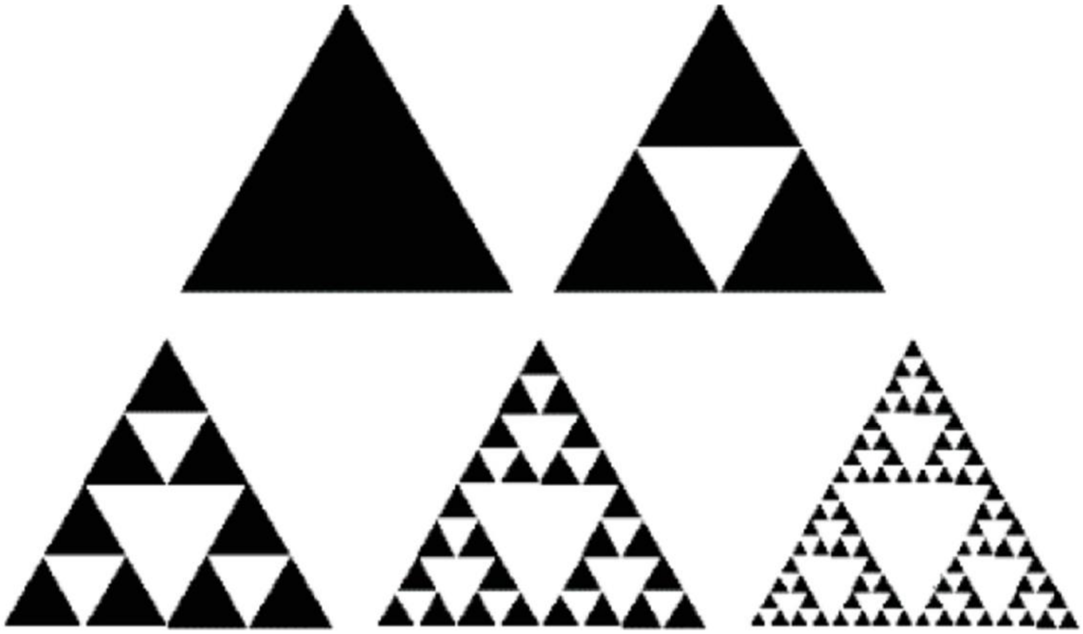


Wikipedia (2023)

Sierpinski Triangle is an iterative process, with the equilateral triangle as the starting point. In the first iteration, the midpoints of each side of this figure are determined and the points are joined. We have four equilateral triangles and remove the one in the middle.

In the second iteration, these steps mentioned in the previous paragraph are repeated for each of the remaining triangles. Thus, this process can be repeated indefinitely, giving rise to the fractal in the figure below.

Figure 2 – Sierpinski Triangle



Wikipedia (2023)

The Menger sponge is a fractal that was first described by the Austrian mathematician Karl Menger (1902-1985), in 1926, based on discussions about its topological dimension.

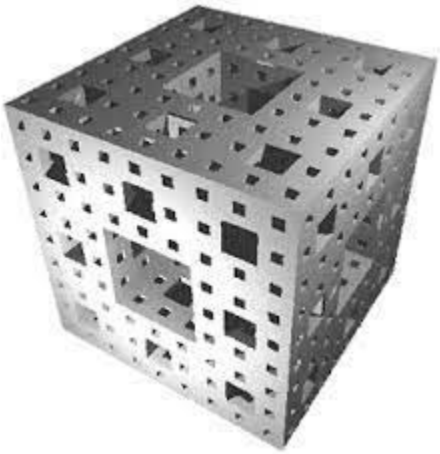
Menger sponge is constructed from a cube , through the following recursive process:

1. In any cube, each face of the cube is divided into 9 squares, thus the cube is subdivided into 27 smaller cubes.

2. The cube located in the middle of each face and the central cube are removed, leaving 20 cubes. We have the first iteration, repeat the process on the remaining 20 cubes and obtain the second iteration at the end of 20 $^{\text{cubes}}$.

Menger sponge is the limit of this process after an infinite number of iterations.

Figure 3 – Menger Sponge



Source: Wikipedia (2022)

The fractal dimension describes how many new pieces geometrically similar to the object are observed when the resolution is increased. If the scale is reduced by a factor F, we find N pieces similar to the original; therefore, the fractal dimension is defined by: N = Fd , where d is the fractal dimension that, in general, would fractionate. Furthermore, it is smaller than the dimension of the Euclidean space in which the fractal lies.

The object F that can be divided into Nr copies of itself on a scale r. For Nr = r $^{-d}$ this relationship can be rewritten in the form:

$$d = -\frac{\log(Nr)}{\log(r)} \quad (1)$$

Following this definition for the fractal dimension of the Cantor Set, we find the value d= (log2) /(log3) = 0.6309 (Figure 1); for Sierpinski Triangle d =(log3) /(log2) = 1.5849 (Figure 2) and for Menger Sponge d = (log20) /(log3) = 2.7268 (Figure 3).

In the case of stochastic fractals, there are several methods for calculating the fractal dimension, among which the most used are: Box -*Counting method* , information dimension and lacunarity.

1.2.1 Area and perimeter dimensions

The perimeter-area method has received several variations over time, but it is mainly used in GIS (Graphic Information System) applications (Halley *et al* ., 2004; Yu *et al* ., 2019) and in landscape ecology, being used as a measure of the complexity of the edges of forest fragments in relation to the area they occupy (Seuront , 2010).

Forest fragmentation is the process of reducing areas of originally connected biotopes, habitats or landscape units into two or more fragments (Lang; Blaschke , 2009). Fragment is a homogeneous element of a landscape and a spatial reference unit for the measurements of its structure.

The shape of a fragment directly affects the relationship between the perimeter and the area of that fragment. Although they are widely applied in landscape ecology, perimeter-area indices are not true dimensions in the mathematical sense (Lopez; Frohn , 2017).

Let P, A and DF, respectively, be the perimeter, area and fractal dimension of the forest fragment, they are related (Krummel *et al* ., 1987):

$$P\alpha\sqrt{A^{D}} \tag{2}$$

where $\propto$ indicates proportionality. Thus, there is a proportionality constant k that makes it true:

$$P = kA^{\frac{D}{2}} \tag{3}$$

applying logarithms to both sides of the equation and developing, we obtain:

$$D = 2.\frac{\ln\left(\frac{p}{k}\right)}{\ln\left(A\right)} \tag{4}$$

The measurement data for the areas and perimeters of the fragments are plotted in a log − log dispersion diagram, linear regression is then applied, since equation 2 , and the fractal dimension is then calculated through the slope of the straight line. regression.

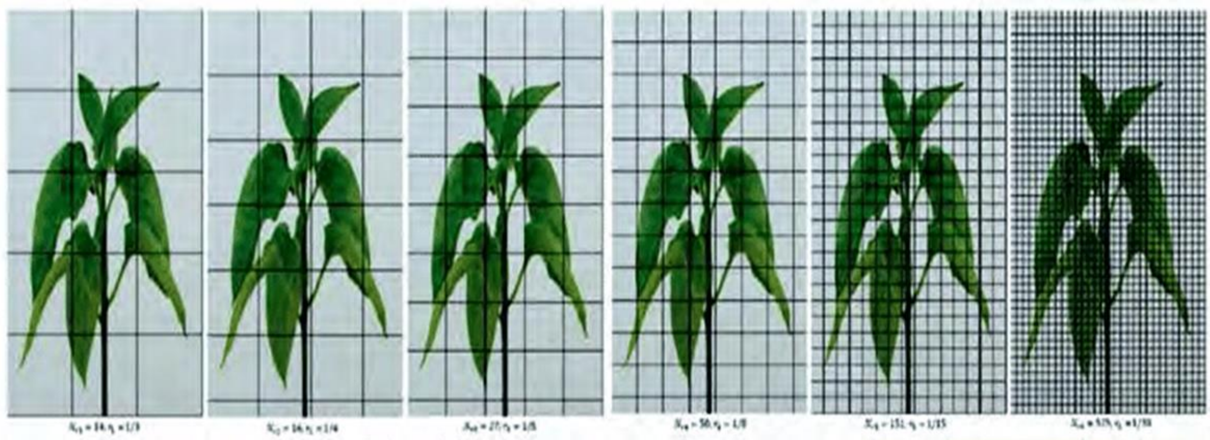
When k = 4, D can be thought to be $D = 2.\frac{\ln(\frac{p}{4})}{\ln(A)}$.

1.2. two Box-Counting Dimension

Being the most used of the fractal dimensions, Falconer (2014) comments that the *Box-Counting dimension* “[...] has a simple intuitive formulation and is one of the most widely used dimensions [...] and its popularity is largely due to partly due to its relative ease of mathematical calculation and empirical estimation" (Falconer , 2014, p. 27, our translation).

The curve, figure or image is covered by a set of objects of the same area or boxes, in this case, square boxes , in which a size is determined for the area of the object and the minimum number of boxes necessary to completely cover the area is counted. figure. As the size of the boxes approaches zero , the total area covered by the boxes converges to the desired length of the curve or image. An example of applying the *Box-Counting method* to calculate the fractal dimension is shown in Figure 4 by varying the scale.

Figure 4 – Application of the *Box-Counting method*



Source: Guerra (2017)

The idea of measurement on different scales is inherent to most of the definitions of fractal geometry, in which different power laws relate measure and scale and, thus, the exponents of these laws define different fractal dimensions, which determine the connection between the space that a object occupies and its size (Rosenberg, 2020; Seuront , 2010). Halley and other authors (2004) state that power laws were already being applied in ecology, even before fractal geometry and that their intimate relationship with fractals increased interest in studies related to fractal geometry. The exponents of the power laws are generally different from unity in biological cases , resulting in graphs in the form of curves, when applied [the power laws] on linear axes, however it is possible to linearize these graphs, through the use of logarithmic axes (Seuront , 2010).

In the context of flat figures, this method consists of dividing the image into boxes, where the number of boxes that contain some part of the figure, which represents the object under study, such as the fractal, are counted and the dimension value can be calculated by following relationship:

$$D_{Box} = \lim_{\varepsilon \to 0} \frac{\ln n(\varepsilon)}{ln\left(\frac{1}{\varepsilon}\right)} \quad (5)$$

Being $\varepsilon$the side of each box and $n(\varepsilon)$the number of boxes that contain some part of the figure.

The smaller the scale, the more precise the established dimension. The idea is basically to measure the figure, disregarding irregularities smaller than the scale ε, analyzing the behavior of the measurement when the scale tends to zero. It is observed that, for three-dimensional objects, boxes are cubes and in n-dimensional spaces they are replaced by hypercubes.

1.2.3 Information dimension

The dimension of information is not a widely used concept, however, in general, it is an approach to thinking about the complexity of information taking into account not only the form, but also the distribution of the information of the object under study. This dimension of information is given by Shannon entropy, defined in Seuront (2010).

The dimension of information with entropy allows us to understand the variety and complexity of the information contained in a set of data. It is an important measure to evaluate the amount of potentially useful and significant information present in a system.

Shannon (1948) proposed a mathematical model to systematize the processing of information in the transmission of messages from one point to another. The Mathematical Theory of Communication (Shannon, 1948) presents a mathematical expression for the amount of information transmitted in a message and provides a way of analyzing the main events that form communication.

To measure the amount of information, Claude Shannon created the concept of entropy. The formula for Shannon entropy (Seuront , 2010) is given by:

$$H(X) = -\sum p(x) \log p(x) \quad (6)$$

Where X is the random variable, associated with some experiment and $p(x)$ is the probability of event x occurring varying all possible events.

In order to adapt the *Box-Counting dimension,* the number of size boxes $\varepsilon$is replaced by the entropy of the size boxes ε, where the probability distribution is given by the frequency of the figure in each box ε, more precisely the Shannon information dimension is given by:

$$D_S = \lim_{\varepsilon \to 0} \frac{\ln H(\varepsilon)}{ln\left(\frac{1}{\varepsilon}\right)} \tag{7}$$

Where:

$$H(\varepsilon) = -\sum_{i=1}^{m(\varepsilon)} \mathrm{p_i}(\varepsilon) \ln\left(\mathrm{p_i}(\varepsilon)\right) \tag{8}$$

being $p_i(\varepsilon)$the relative frequency of the object in the box of size $\varepsilon$and *m* is the number of boxes of size ε . In other words, given N the total number of pixels in the image and $f_i(\varepsilon)$ a the number of pixels, in the size box ε, we have:

$$p_i(\varepsilon) = \frac{f_i(\varepsilon)}{N} \tag{9}$$

Varying $i$the number of boxes of size ε, the consequence is that $\sum_{i=1}^{m} p_i(\varepsilon) = 1$.

However, here we also propose the use of Tsallis entropy , with the aim of highlighting the boxes that have more information about the object. The successful applications of Tsallis entropy have motivated investigation of the mathematical tool behind this structure.

The role of statistical mechanics, by Tsallis , was to propose a possible generalization of the so-called Entropy of Boltzmann, Gibbs and Shannon, which would enable a way of describing physical systems (Bekenstein , 1973).

Bearing in mind that the Shannon information dimension takes into account the amount of information in each box, it becomes quite natural to generalize the idea by applying Tsallis entropy defined by:

$$H_{TS}(\varepsilon) = \frac{1}{q-1}\left(1 - \sum_{i=1}^{m} p_i(\varepsilon)^q\right) \tag{10}$$

Emphasizing that, in the limit, when , $q \to 1$Tsallis entropy assumes the value of Shannon entropy. The Tsallis information dimension is then defined by:

$$D_{TS} = \lim_{\varepsilon \to 0} \frac{\ln H_{TS}(\varepsilon)}{ln\left(\frac{1}{\varepsilon}\right)} \tag{11}$$

The parameter $q$leverages the information in each size bin $\varepsilon$and this can be used to better distinguish boundary disturbances.

1.2.4 Lacunarity

As a complement to the fractal dimension, Mandelbrot (1983) introduced the concept of lacunarity (a word derived from the Latin “lacuna”, which means “empty”) to describe and quantify the deviation of fractal objects from their translational invariance, evaluating the size distribution of gaps in object texture.

Newman and other authors define lacunarity as “[...] a way of characterizing the spatial configuration of points or other components of a spatial pattern, such as fragments or pixels” (Newman *et al.,* 2019, p. 10, our translation). Rosenberg (2020), in turn, defines lacunarity as a measure of the uniformity of a set.

While the fractal dimension is calculated through the distribution of the object in space, quantifying the size and this measurement changes at different scales, lacunarity is measured by analyzing the distribution of empty spaces (gaps) left by the object.

The properties and characteristics of a fractal are not completely determined by its size. Because lacunarity is a distinct and independent feature of the fractal dimension, it strongly relates to the distribution and size of holes in the fractal and uses probability moments of order one and two.

The lacunarity index leads to the calculation of the probability moments of an image and this can be considered as the sum of the moments of each of its mass points. The mass distribution is given by P:

$$P(S, r) = \frac{n(S,r)}{N(r)} \tag{12}$$

Where n (S, r) is the number of boxes containing exactly S image points and N(r) the total number of boxes on side r.

By definition P we have $\sum P = 1$, with P being a probability distribution, allowing moment 1 and 2:

Z 1 (r) = $\sum_{s=0}^{r^2} s \ * P(s,r)$and Z 2 (r)=$\sum_{s=0}^{r^2} s^2 * P(s,r)$

The lacunarity on the r scale is given by the moment quotient:

$$L(r) = \frac{Z2}{(Z1)^2} \tag{13}$$

CHAPTER II. MATERIALS AND METHODS

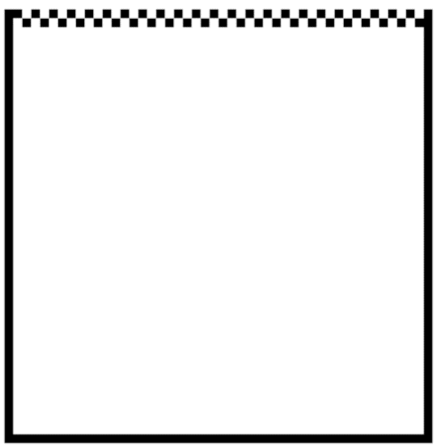
2.1 DATA AND *SOFTWARE* USED

In this study, squares with small boundary disturbances, with the same mass, built in the GIMP *software* , translation version 2.10.34, were used as data, and also the same for creating figures with different patterns and pixel distributions. The algorithm was implemented to calculate the dimension of information and the lacunarity that plotted the graphs and enabled the creation of tables in Python programming language, this occurred in the integrated development environment of VSCode , where it presented good computational performance for the information dimension.

2.2 IMPLEMENTATION OF THE *BOX-COUNTING DIMENSION*

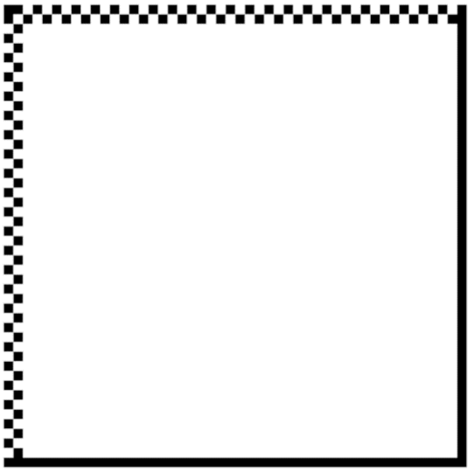
software was used to construct squares, as in Figures 5, 6, 7 and 8, with contours of the same mass, that is, the same number of pixels on their border, with the peculiarity that their vertices and dimensions were powers of two. This concern is that subdivisions of the boxes in the *Box-Counting algorithm* would maintain the intersections of the previous division.

Figure 5 – Square A with 1 disturbed side



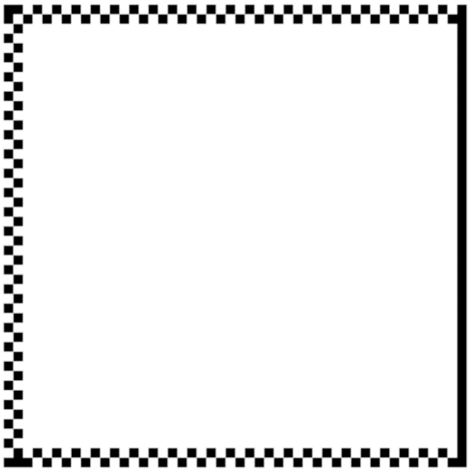
Source: The author (2023)

Figure 6 – Square B with 2 disturbed sides



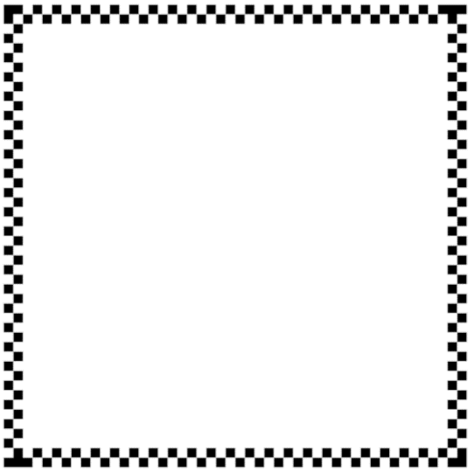
Source: The author (2023)

Figure 7 – Square C with 3 perturbed sides



Source: The author (2023)

Figure 8 – Square D with 4 perturbed sides



Source: The author (2023)

software called FracDim was also developed in the Python language to calculate the linear regression of the *Box-Counting dimensions.* The dimension will then be the slope of the regression line associated with the logarithm values, that is, the size of the side of the box and the log of the number of boxes.

Initially, after implementing the *Box-Counting dimension,* tests were carried out with regular geometric figures and with each advance, subtle changes were generated in the figure's borders to obtain different flat figures with the same *Box-Counting dimension* . Arriving, then, at the modified squares on one, two, three and four sides, represented in the aforementioned figures, which have the same *Box-Counting dimension* and do not have the same Information dimension.

2.2.1 Box-Counting Algorithm

Box-Counting algorithm consists of covering the fractal by N (ε) boxes of size εthat contain at least one point of the object. The process is repeated with boxes of different sizes and a graph of ln [N ( )] $\varepsilon$is plotted as a function of $\ln(\varepsilon)$.

Figure 9 – Algorithm for the *Box-Counting dimension*

```
def CalculoDimensaoBoxCounting():

    global PontosDaBoxCounting
    x0=0
    y0=0

    WF=Img.width
    HF=Img.height
    #WF= Paisagem.winfo_width()
    #HF= Paisagem.winfo_height()
    print(WF,HF)
    global DesenhaBox
    x=[]
    y=[]
    for Passos in range(1,9):
        if Passos==8:
           DesenhaBox=true
        DI=2**Passos # número de divisões
        W=(WF/DI) # largura dos box's
        H=(HF/DI) # altura dos box's
        NumeroDeBox=ContaBoxs(x0,y0,W,H,WF,HF)
        x.append(ln(1/min(W,H))) # W = H na figura quadrada
        y.append(ln(NumeroDeBox))
    PontosDaBoxCounting.append(x)
    PontosDaBoxCounting.append(y) # Guarda pontos para plotagem dos gráficos
    return(x,y)
```

Source: The author (2023)

The algorithm scans and counts the boxes that contain the figure, where x is the ln (1/min(W,H), W is the width of the box and H is the height of the box, and in the y coordinate, the number of boxes that covers the figure.

The countBoxs function calls the parameters (x0, y0, W, H, WF, HF) responsible for counting the number of boxes that contain the figure.

Figure 10 – Algorithm for ContaBoxs

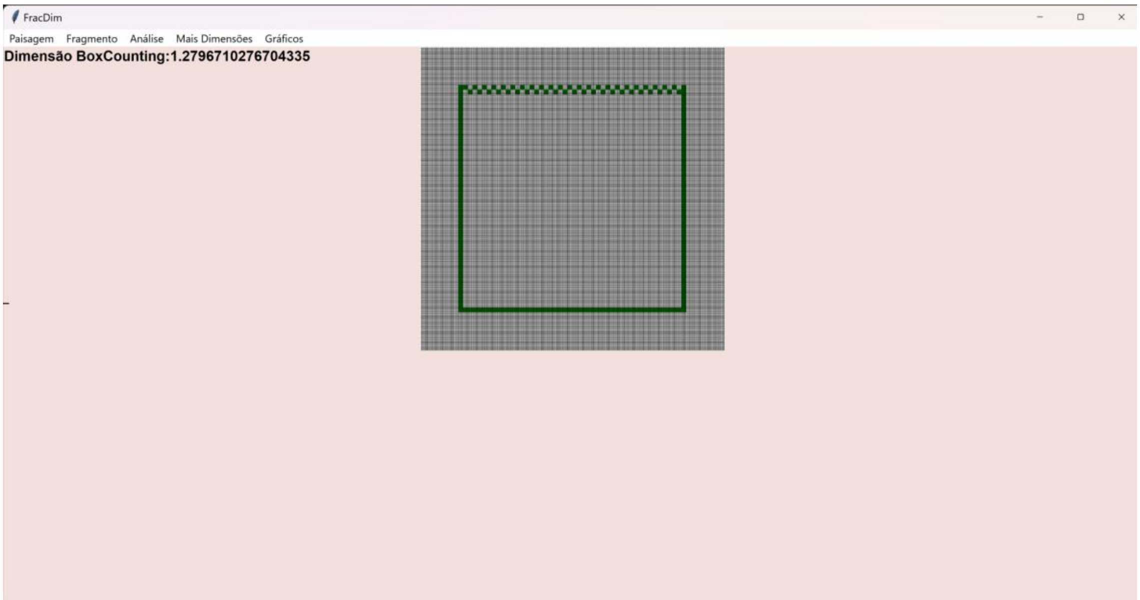
```
def ContaBoxs(x0,y0,W,H,WF,HF):

    DH=0
    DW=0
    cont=0
    while(DH*H<HF):
        while(DW*W<WF):
            if (TestaSeBoxIntersectaFigura(x0+DW*W, y0+DH*H, x0+(DW+1)*W, y0+(DH+1)*H)):
                cont=cont+1
            DW=DW+1
        DH=DH+1
        DW=0
    return(cont)
```

Source: The author (2023)

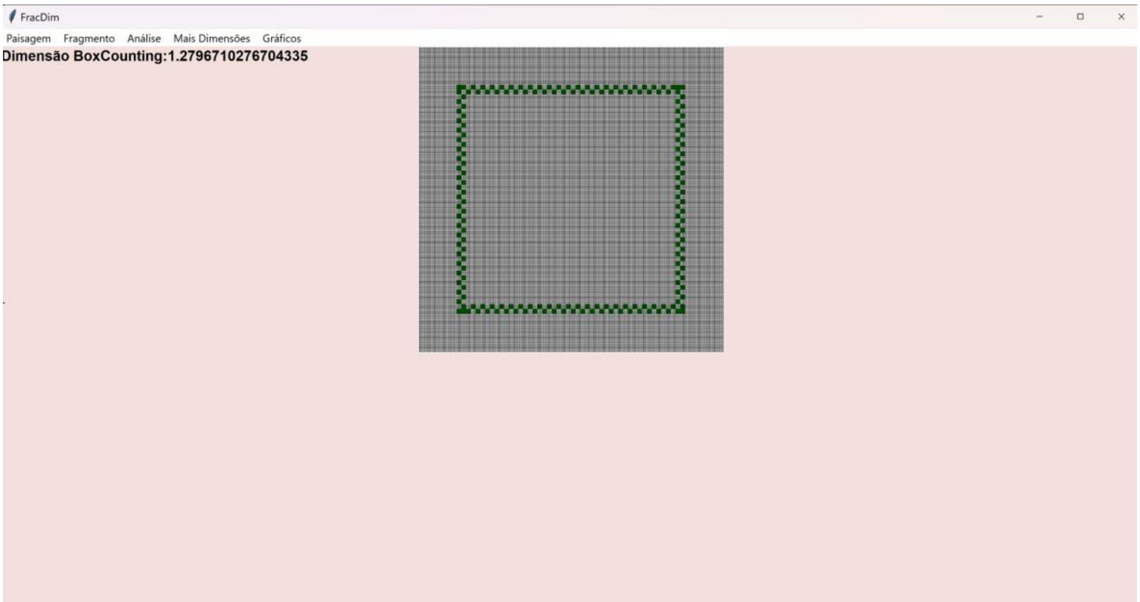
Figures 11 and 12 show in *green* the boxes that cover the Figure, the number of divisions, that is, the power is linked to the number of steps of the function, height and width of the boxes, generating a *Box-Counting dimension* , which is displayed on the top left side of the screen.

Figure 11 – *Box-Counting* result in the square with 1 perturbed side



Source : The author (2023)

Figure 12 – *Box-Counting* result in the square with the 4 disturbed sides



Source : The author (2023)

2.3 IMPLEMENTATION OF THE INFORMATION DIMENSION

Box-Counting dimension were then constructed , but that do not have the same information dimension, thus proving that the information dimension resolves ambiguities generated by the *Box-Counting dimension* and reveals more about the complexity of the boundary of certain flat objects.

From the dimension of the implemented information, which facilitates the collection of results by detecting disturbances , it is possible to study the concomitant dimension in the development of Shannon and Tsallis entropy calculations . And this can be widely applied to solve problems with modeling in architecture, in medicine, such as heart rate, in degradation detection in the field of ecology, or material wear, in the field of Engineering.

2.3.1 Information Dimension with Shannon entropy

Entropy is a measure of the uncertainty or disorder present in a system. When applied to the information dimension, entropy can be considered as a measure of the amount of information contained in a set of data or how the information is distributed within that set.

Historically, entropy arose in information theory to calculate the amount of average information needed to represent or encode a message in a given system. It is calculated based on the probability of occurrence of different events or symbols in a set of data.

The central concept of entropy was created to measure the expected rarity or surprise of a random variable X in its distribution. In literature, entropy is generally considered as information. It quantifies the average information needed to describe a set of data, considering the probability distribution of events.

Shannon entropy is an interpretation of results through the quantification of information. In this way, the opportunity arises to measure disorganization, or uncertainty, and use it as image information, which will serve as a classification criterion. Entropy can be understood as a mathematical measure, which defines the amount of information necessary to describe a random variable, or as a measure of uncertainty about it.

2.3.1.1 Information dimension algorithm with Shannon entropy

In the study, the data set for entropy is the result of the number of *pixels* distributed in each box and the dimension of information with entropy refers to the amount of different and distinct information that is present in a data set. The greater the entropy, the greater the variety and diversity of information contained in the set.

Figure 13 – Algorithm for calculating dimension with Shannon entropy

```
def CalculoDimensaoDaInfShannon():

    global PontosDeShannon
    x0=0
    y0=0
    WF= Paisagem.winfo_width()
    HF= Paisagem.winfo_height()
    global DesenhaBox
    x=[]
    y=[]
    for Passos in range(1,9):
        if Passos==8:
           DesenhaBox=true
        DI=2**Passos # númnero de divisões
        W=(WF/DI) # largura dos box's
        H=(HF/DI) # altura dos box's
        EntropiaShannon=Entropia(x0,y0,W,H,WF,HF)
        x.append(ln(1/min(W,H))) # W = H na figura quadrada
        y.append(ln(EntropiaShannon))
    PontosDeShannon.append(x)
    PontosDeShannon.append(y) # Guarda pontos para plotagem dos gráficos
    return(x,y)
```

Source: The author (2023)

The Shannon entropy in the algorithm calls the parameters, the function contaPixelDaFigureNoBox , counts the boxes of the figure that contains pixels, n_delta , as well as checks the frequency in p_delta , calculating the entropy H_delta .

Figure 14 – Shannon entropy algorithm

```python
def Entropia(x0,y0,W,H,WF,HF):

    DH=0
    DW=0
    n_delta=[] # número de pixel da imagem no box delta
    p_delta=[] # ´probabilidade dada pela frequência de pixels na caixa delta
    H_delta=0
    while(DH*H<HF):
        while(DW*W<WF):
            cont=ContaPixelsDaFiguraNoBox(x0+DW*W, y0+DH*H, x0+(DW+1)*W, y0+(DH+1)*H)
            if cont!= 0:
                n_delta.append(cont)
            DW=DW+1
        DH=DH+1
        DW=0
    N=0 # N será o número total de pixels da figura
    for i in range(len(n_delta)): # Calcula o total de Pixel
        N=N+n_delta[i]
    for i in range(len(n_delta)): # Calcula afrequência em cada caixa
        p_delta.append(n_delta[i]/N)
    for i in range(len(n_delta)): # Calcula a entropia da divisão delta
        H_delta=H_delta+p_delta[i]*ln(1/p_delta[i])
    return(H_delta)
```

Source: The author (2023)

Tsallis entropy

Constantino Tsallis had been arguing that entropy needed refinement to cover many types of systems, since for Tsallis , entropy works perfectly even for certain limits. Therefore, in 1988, the generalization of statistics was first proposed by Constantino Tsallis , generalization of Boltzmann's entropy, in his article entitled “ Possible Generalization of Boltzmann-Gibbs Statistics " (Tsallis , 1998).

Boltzmann (Shannon) entropy has its limitations, it even works for certain parameters, but here, a non-extensive entropy will be presented, which depends on a parameter *q,* whose name is called the Tsallis entropic index – a way of characterizing correlations of a system, the interdependence of two or more variables, particularly how strong they are. This entropy has been successfully applied to explain the behavior of non-extensive complex systems.

Furthermore, the parameter $q$enhances the information in each size bin $\varepsilon$and this can be used to better distinguish boundary disturbances.

Tsallis entropy :

Table 1 – Tests with the square varying the entropic index q

	Square varying the index value
index q	Tsallis entropy value
q = 0.4	27.03771
q = 0.5	15.15823
q = 0.9	1.89600
q = 0.99	1.25488
q = 0.999	1.20549
q = 2	0.03943

Source: The author (2023)

The calculation of the information dimension with Tsallis entropy is an important aspect of the method, as the choice of the parameter q , which is of great value in the dimension, thus after an analysis of Table 1, allowed us to determine the complexity of the signals with q = 2, entropic index remained in the range 0.03, in the other values for q between 0.9 and 0.999 in the results obtained we can observe that the dimension in the algorithm remained in the range between 1 and 2.

Tsallis entropy , a good option for its implementation.

2.3.2.1 Information dimension algorithm with Tsallis entropy

Tsallis entropy are defined and the parameters are placed, including q, entropic index q = 0.5, which produce more effective estimates proven in the tests.

Figure 1 5 – Algorithm for information dimension with Tsallis entropy

```python
def DimensaoDaInfTSallis():

    x,y=q_DimDaInfDeTSallis(0.5)
    a,b=np.polyfit(x,y,1)
    Texto="Dimensão Da Informação de TSallis:"+str(a)
    Legenda.config(text=Texto)
    Legenda.place(x=0,y=0)
    app.mainloop()


def q_DimDaInfDeTSallis(q):

    global PontosDeTSallis
    x0=0
    y0=0
    WF= Paisagem.winfo_width()
    HF= Paisagem.winfo_height()
    global DesenhaBox
    x=[]
    y=[]
    for Passos in range(1,9):
        if Passos==8:
            DesenhaBox=true
        DI=2**Passos # número de divisões
        W=(WF/DI) # largura dos box's
        H=(HF/DI) # altura dos box's
        EntropiaTS=EntropiaTSallis(q,x0,y0,W,H,WF,HF)
```

Source: The author (2023)

Tsallis Entropy, the parameters are determined and x = ln (1/min (W, H)) and oy = ln of Tsallis entropy are also defined . Below the code presents the Tsallis entropy , for the vectors n_delta , p_delta and zera Sp_delta , the sum of the frequencies. In Figure 16, ' cont ' represents the countpixel of the figure, after scanning all the boxes we have the dimension for this entropy.

Figure 1 6 – Algorithm for Tsallis entropy

```python
def EntropiaTSallis(q,x0,y0,W,H,WF,HF):
    x0=0
    y0=0
    WF= Paisagem.winfo_width()
    HF= Paisagem.winfo_height()
    DH=0
    DW=0
    n_delta=[]
    p_delta=[]
    Sp_delta=0
    while(DH*H<HF):
        while(DW*W<WF):
            cont=ContaPixelsDaFiguraNoBox(x0+DW*W, y0+DH*H, x0+(DW+1)*W, y0+(DH+1)*H)
            if cont!=0:
                n_delta.append(cont)
            DW=DW+1
        DH=DH+1
        DW=0
    N=0
    for i in range(len(n_delta)):
        N=N+n_delta[i]
    for i in range(len(n_delta)):
        p_delta.append(n_delta[i]/N)
    for i in range(len(n_delta)):
        Sp_delta=Sp_delta+p_delta[i]**q #soma das frequências elevadas a q
    S_delta=1/(q-1)*(1-Sp_delta)
```

Source: The author (2023)

CHAPTER III. GAP

3.1 DEFINITION

Mandelbrot (1983) noted that the fractal dimension is not sufficient for a complete characterization of the texture of objects with fractal properties. There are fractal objects with the same fractal dimension and with different appearance in their texture (Dong, 2000).

Lacunarity is a scale-dependent measure, as structures that are homogeneous on small scales can be heterogeneous on larger scales and vice versa (Plotnick ; Gardner; O'Neill, 1993) . Different fractals can have the same fractal dimension, while having different lacunarity measures.

Although dimensions provide valuable information about the geometry of objects, the fact that objects of completely different shapes share the same fractal dimension makes it necessary to have other methods of analyzing fractal objects (Barnsley *et al* ., 1988).

3.2 IMPLEMENTATION OF THE GAP

Depending on the value of the lacunarity, the fractal object is classified as being more or less homogeneous. It is expected that in these Figures, the patterns that present gaps of the same size, with a more homogeneous mass distribution, show the lowest lacunarity index, unlike figures that have gaps of varying sizes. Below are the figures used to implement the lacunarity:

Figure 17 – Standards used to calculate the lacunarity index, with the same mass

Source : The author (2023)

3.2.1 Mathematical model

There is no single method for calculating the lacunarity of fractal objects, but they all basically measure their mass distribution. Used the sliding box method assigned in the FracDim application, a method that is well known and easy to implement.

If the image is digital, the lacunarity measures its pixel distribution as shown in figure 18 in the code with the contapixelnoBox function .

A box of size r is placed at the origin of the FracDim screen, starting from the center and the number of boxes with occupied mass S is counted. The box is moved across the entire set of observations, and its mass is calculated.

The lacunarity is defined as a function of P (S, r), which is the fraction of windows of radius r, in the square format which is easier to implement and with mass S that the shape contains, that is, representing the number of boxes occupied with mass "S" which is counted.

The box is moved across the entire set of observations and its mass is calculated. This process is repeated for the entire set of observations, obtaining the frequency distribution of the mass of box n (s, r), and correspondingly the probability distribution P(s,r) = n(s,r)/N(r), where N(r) is the total number of boxes of size r slid on the desktop.

By definition P we have ∑P = 1, with P being a probability distribution, allowing moments 1 and 2. The lacunarity on the r scale is given by the moment quotient.

3.2.2 Algorithm for calculating lacunarity

The lacunarity, initially, in the code, defines the BoxSliding function , determining the global Landscape to open the pattern, the boxes slide (C, r), varying the C = center and or = radius.

Figure 1 8 – BoxSliding and ContaPixelDoBox function defined

```python
def BoxDeslizante(C,r):

    global Paisagem

    x1=C[0]-r
    x2=C[0]+r
    y1=C[1]-r
    y2=C[1]+r
    Paisagem.create_rectangle(x1, y1, x2,y2,outline="green", tags="ret")
    app.update()

def ContaPixelsDoBoxr(xc,yc,r):

    global Paisagem
    global Img
    ContPixel=0
    x1=xc-r
    x2=xc+r
    y1=yc-r
    y2=yc+r
    for i in range(x1+1,x2-1):
            for j in range(y1+1,y2-1):
                if Img.getpixel((i,j))!=255:
                    ContPixel=ContPixel+1
    return(ContPixel)
```

Source: The author (2023)

In the function contapixelDoBox (Figure above), the parameters x and y of the center are defined xc and yc important for the lacunarity, which is defined as a function of P (S, r), which is the fraction of windows of radius r with mass S that the shape contains, that is, representing the number of boxes occupied with mass “S”, which is counted, generating the function NumeroDeBoxcomMassaS .

Figure 19 – Algorithm of the NumDeBoxcomMassaS function

```python

def NumeroDeBoxComMassaS(S,r):

    Cont=0
    for xc in range(r+1,128-r): # xc é o x do centro da janela
        for yc in range(r+1,128-r):  # yc é o y do centro da janela

            if ContaPixelsDoBoxr(xc,yc,r)==S:
                Cont=Cont+1
    return(Cont)

def TotalDePixelsDaImg():
    global PixelsDaImg
    cont=0
    for i in range(128):
        for j in range(128):
            if Img.getpixel((i,j))!=255:
                cont=cont+1
    PixelsDaImg=cont


```

Source: The author (2023)

Finally, in the algorithm d the lacunarity index, the moments are defined, with a working area l = 128, the number of boxes with mass S is given and the lacunarity index is calculated by applying the quotient of the moments.

Figure 20 – Algorithm to define the moments in lacunarity

```
def Momentos(r):
    l=128 #  Tamanho da área de trabalho
    Z1r=0
    Z2r=0
    Nr=(l-2*r)**2

    for S in range(1,(1+2*r)**2):
        Q_Sr=NumeroDeBoxComMassaS(S,r)/Nr
        Z1r=Z1r+S*Q_Sr
        Z2r=Z2r+(S**2)*Q_Sr

    return(Z1r,Z2r)
```

Source: The author (2023)

Figure 21 – Gap algorithm

```
def Lacunaridade(r):
    Zr=Momentos(r)
    Z1r=Zr[0]
    Z2r=Zr[1]

    if Z1r!=0 :
        return(Z2r/(Z1r**2))
    else:
        return(0)
def CalculaLacunaridade():

    x=[]
    y=[]

    for r in range(1,100):
        print(r)
        x.append(r)
        y.append(Lacunaridade(r))

    plt.plot(x,y)
    plt.xlabel("r")
    plt.ylabel("$\Lambda(r)$")
    plt.title("Lacunaridade")
    plt.show()
```

Source: The author (2023)

CHAPTER IV. FRACDIM APP

4.1 MENU AND APPLICATION FUNCTIONS

Initially, this application was developed for analyzing images of landscape fragments and was called FracFor . The application so far has five menus with different functionalities (Figure 21) already in operation.

Open, Save, Close and *Exit* functions . The *Open* icon works with images in JPG, PNG or TIFF formats and can be loaded for editing. The *Save* function saves this first image file and a spreadsheet with fragment data.

New function allows you to outline the fragment by defining vertices of a polygon, while the *Delete* function erases the outlined polygon and *Select* chooses the polygon for analysis.

More Dimensions and Graphics menu bar was added . In *More Dimensions the functions appear: Box-Counting* Dimension , Shannon Information Dimension, Tsalli Information Dimension for q = 0.5 . While the *Charts* menu generates charts of dimensions and measurement results. In the *Chart* menu , you will find the options:

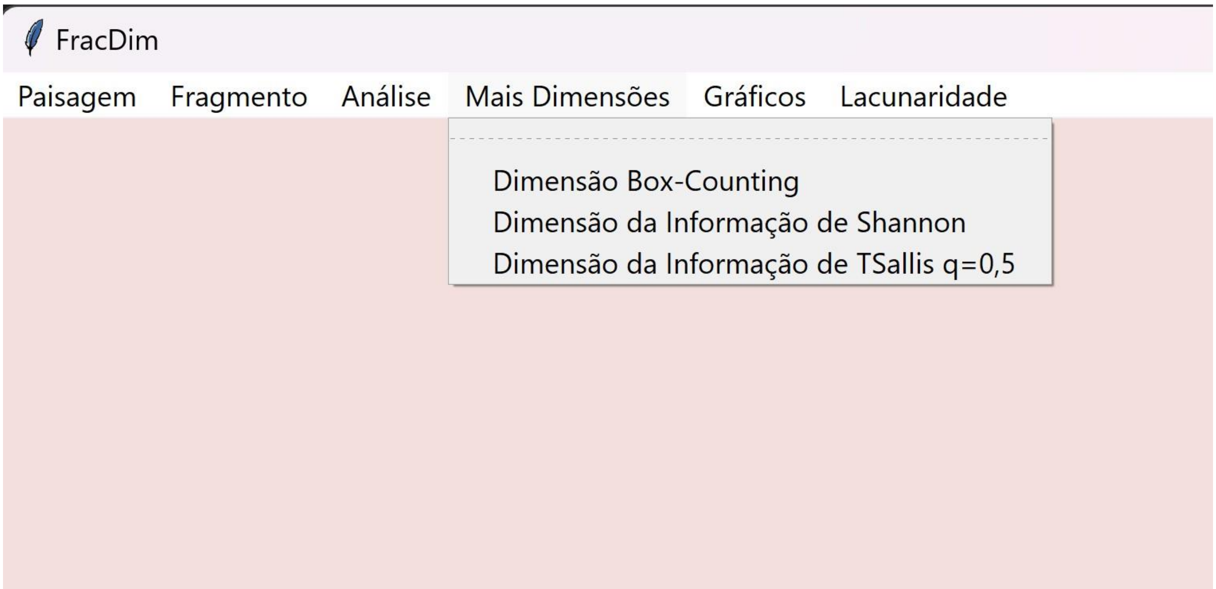
- ✓ *Box-Counting* dimension graph , which can be polygonal or linear regression;
- ✓ information dimension graph with Shannon entropy, also polygonal or regression;
- ✓ the two on the same graph, *Box-Counting* and Shannon, used in part of the study to compare;
- ✓ information dimension graph with Tsallis entropy for q = 0.5;
- ✓ and finally, the information dimension graph with Tsallis entropy .

The last menu is the calculation of lacunarity, this already plots the graph.

For the algorithm to work, a folder called FracDim was organized, which contains important elements for its operation. The functions, some described in this research, in Figures 9 and 13 and others that show part of the code and the def = function defines the algorithm function, in addition to the variables and packages that are other elements that appear in the folder. All commands in the FracDim.py folder are in the Python language.

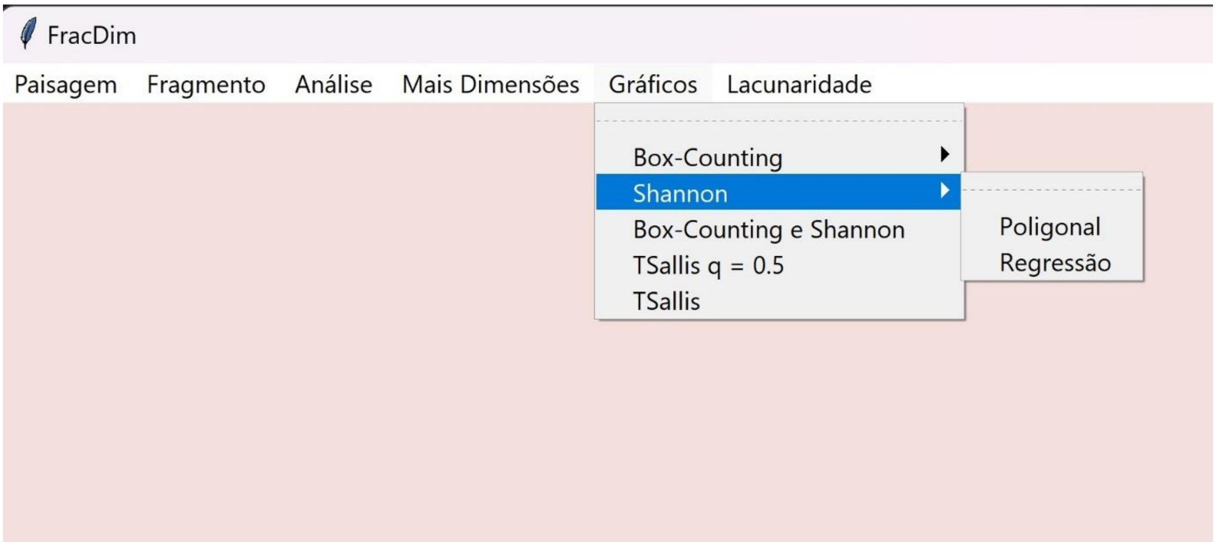
4.2 IN PRACTICE

Figure 22 – The fractal dimensions in the Menu bar of the FracDim application



Source: The author (2023)

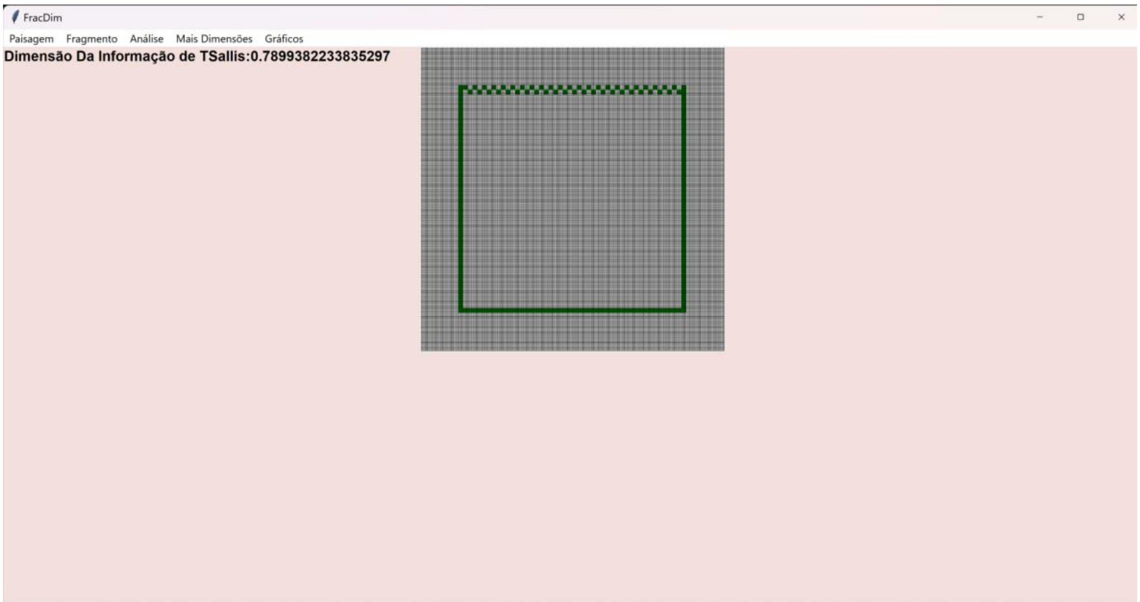
Figure 23 – FracDim Menu Bar in the Graphics function



Source: The author (2023)

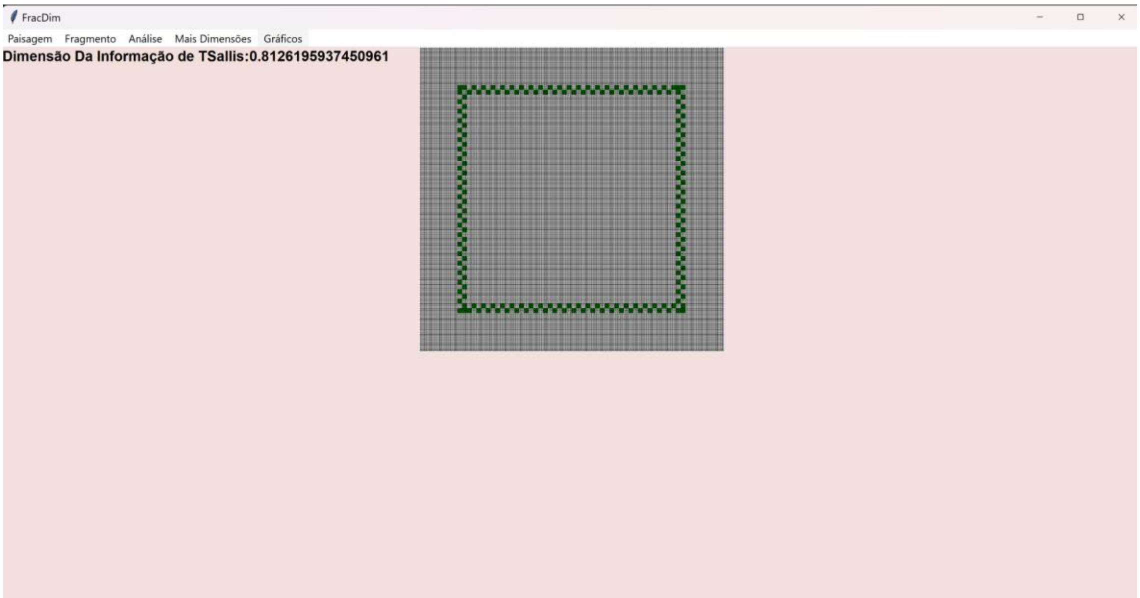
Figures 24 and 25 show the FracDim application in use to calculate the information dimension with Tsallis entropy .

Figure 24 – Result of the Information Dimension with Tsallis entropy , in square A with 1 disturbed side



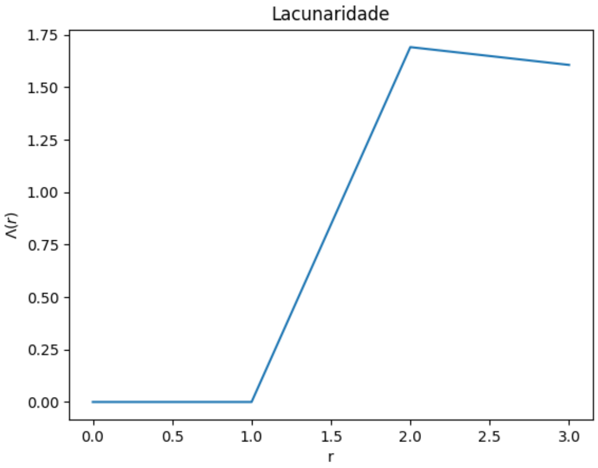
Source : The author (2023)

Figure 25 – Result of the Information Dimension with Tsallis entropy , in square D with the 4 disturbed sides



Source : The author (2023)

Figure 26 – Test graph in the Fracdim application of the lacunarity index with radius < 10 with standard C



Source: The author (2024)

In the FracDim application for calculating lacunarity, the sliding box is moved across the entire image and its mass is calculated. This process is repeated for the entire set of observations, obtaining the frequency distribution of the box mass n (s, r), and correspondingly the probability distribution P (s, r). The total number of boxes of size r slid on the work area l = 128 is in probability and as the radius increases, the value of the average over these increases and the lacunarity decreases until the limit at which the box captures the shape.

In the lacunarity code, the quotient of moments is implemented, moment 2 per moment 1 squared, then the x and y axes are defined, in the range the radius (1;100) which was the objective for a clearer graphical representation that provided a better reading of the lacunarity as a function of r. Then, FracDim plots the graph.

CHAPTER V. RESULTS AND DISCUSSION

5.1 ANALYSIS OF *BOX-COUNTING RESULTS* AND INFORMATION DIMENSION WITH SHANNON ENTROPY

Examples of geometric figures with small boundary perturbations were generated, where one can observe the application of the information dimension in resolving the ambiguities of the *Box-Counting dimension* . Both Shannon's and Tsallis ' information dimensions apply .

This ambiguity in the *Box-Counting Dimension* is presented in Figures 5 to 8, in which the squares were built with the same mass and presenting different borders. In Figure 5 the measurement is generated for the square with 1 perturbed side and the measurement presented in FracDim is D_{Box} = 1.2796710276704335, while in Figure 8 the square has all 4 sides disturbed and the same measurement is obtained, that is, the index presented ambiguity for the *Box-Counting dimension* .

Table 2 demonstrates, in the second column, the ambiguity of the *Box-Counting dimension* , $D_{Box}$by obtaining the same values for squares A, B, C and D. Furthermore, in the third column, D_SShannon's information dimension:

Table 2 – Result of the *Box-Counting* Dimension and the Information Dimension with Shannon entropy

Square	$\boldsymbol{D_{Box}}$	$\boldsymbol{D_S}$
A	1.2796710276704335	0.33592615925964914
B	1.2796710276704335	0.33781554676065440
W	1.2796710276704335	0.33965496788217200
D	1.2796710276704335	0.34131330650656094

Source : The author (2023)

The Information dimension provides more details, as unlike the *Box-Counting dimension,* which only counts the boxes that contain the image, DI has the number of pixels or points within each box and this is expressed as the relative frequency.

Thus, a weight is assigned to each box and those with a higher number of points count more than those with a lower number. When applied to the equation, it presents the greatest results, a measure of complexity that detects the disturbance in the figure.

5.1.1 Demonstration

It is also important to note in the graphs in Figures 5, 6, 7 and 8, corresponding to the disturbed squares A, B, C and D, that the Shannon information dimension is always smaller than the *Box-Counting dimension,* considering that in all graphical representations, DI is smaller than D_{Box}, it is possible to mathematically prove that this is true. Observing the inequality below, we intend prove that:

INFORMATION DIMENSION (DI) ≤ *BOX-COUNTING DIMENSION* (D_{Box}), where:

$$DI = \lim_{\varepsilon \longrightarrow 0} \frac{\ln [H(\varepsilon)]}{\ln (\frac{1}{\varepsilon})}$$

$$D_{Box} = \lim_{\varepsilon \longrightarrow 0} \frac{\ln [n(\varepsilon)]}{\ln (\frac{1}{\varepsilon})}$$

In general, the entropy of possibilities forms the sample space X= { x_1 , x_2 ,... , x_4 } and its probabilities P_i = P (x_i) = probability of x_i occurring . Thus, we obtain:

$$\sum_{i=1}^{n} \mathrm{p_i} = 1$$

It is known that Shannon Entropy is calculated by:

$$H(\varepsilon) = -\sum_{i=1}^{n(\varepsilon)} \mathrm{p_i}(\varepsilon) \ln\left(\mathrm{p_i}(\varepsilon)\right)$$

In information entropy, for each box , the frequency of the image can be determined as follows:

$$fi(\varepsilon) = \frac{mi(\varepsilon)}{N},$$

for mi = number of pixels in the figure in the box i and N= total number of pixels in the figure. That way:

$$H(\varepsilon) = -\sum_{i=1}^{m(\varepsilon)} fi(\varepsilon) * \log[fi(\varepsilon)]$$

$$\mathbf{H(\varepsilon)} = \sum fi \ln(1/fi) \leq \sum fi \cdot \frac{1}{fi} \longrightarrow \sum 1 = \mathbf{n(\varepsilon)} \qquad \mathbf{(1)}$$

To prove that the Information Dimension is less than or equal to the *Box-Counting dimension* , we must:

$$DI \leq D_{Box}$$

$$\frac{\ln[H(\varepsilon)]}{\ln(\frac{1}{\varepsilon})} \leq \frac{\ln[n(\varepsilon)]}{\ln(\frac{1}{\varepsilon})}$$

By (1), we have:

$H(\varepsilon) \leq n(\varepsilon)$

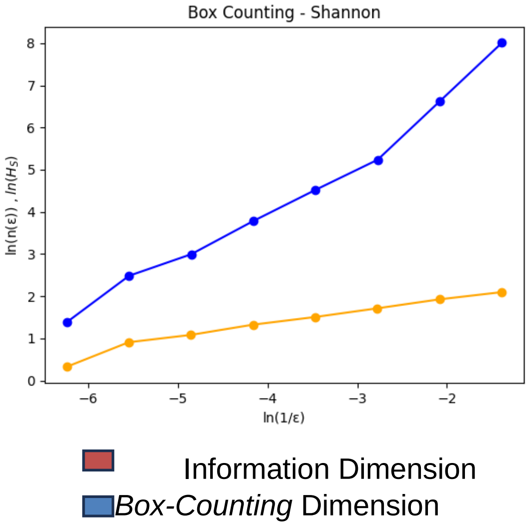
$\ln[\mathbf{H(\varepsilon)}] \leq \ln[\mathbf{n(\varepsilon)}]$, then:

$$DI = \lim_{\varepsilon \longrightarrow 0} \frac{\ln[H(\varepsilon)]}{\ln(\frac{1}{\varepsilon})} \leq \lim_{\varepsilon \longrightarrow 0} \frac{\ln[n(\varepsilon)]}{\ln(\frac{1}{\varepsilon})} = D_{Box}.$$

5.1.2 Graphs of the Box-Counting Dimension with the Information Dimension with Shannon entropy

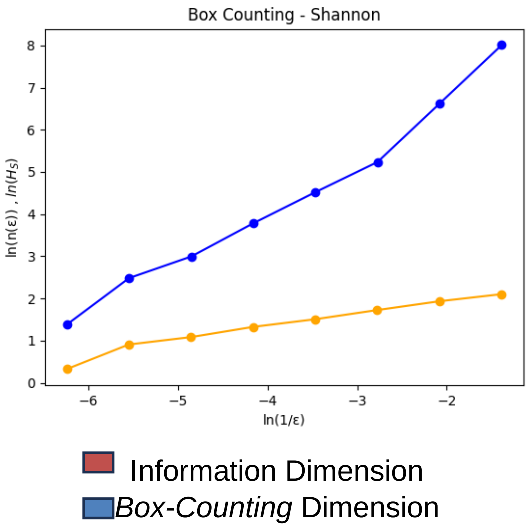
In such graphs, $y$the Neperian logarithm of the number of boxes that intersect the figure and the entropy of each of the boxes is represented on the axis, while along the axis $x$the Neperian logarithm of $\frac{1}{\varepsilon}$, where $\varepsilon$is the length of the side of each box.

Figure 27 – *Box-Counting* Dimension Chart and Information Dimension with Shannon entropy for square A



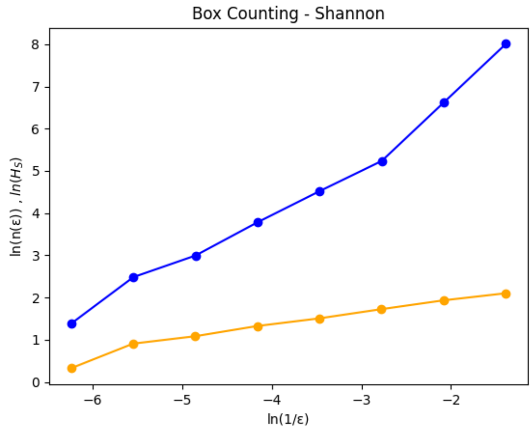
Source : The author (2023)

Figure 28 – *Box-Counting* Dimension Chart and Information Dimension with Shannon entropy for square B



Source : The author (2023)

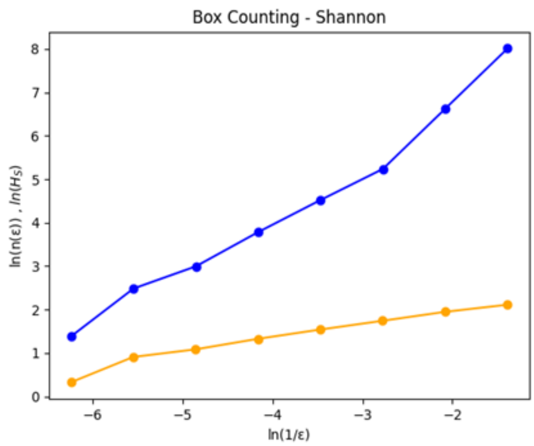
F igure 29 – *Box-Counting* Dimension Graph and Information Dimension with Shannon entropy for the C square



Information Dimension
Box-Counting Dimension

Source : The author (2023)

Figure 30 – *Box-Counting* Dimension Graph and Information Dimension with Shannon entropy for the D square



Information Dimension
Box-Counting Dimension

Source : The author (2023)

5.2 DIMENSION OF INFORMATION WITH TSALLIS ENTROPY

Tsallis entropy was introduced into the Information Dimension index, which allowed greater emphasis on the knowledge contained in each box, due to its entropic parameter q. This parameter made it possible to modulate sensitivity to information as shown in Figures 24 and 25.

The concept of entropy identifies when a data set is composed of only a few symbols, making entropy low, as there is little variety of information. On the other hand, in a dataset with a uniform distribution of different symbols, entropy would be high as there is a greater diversity of information.

The application reveals results where the borders of squares, with 1 side perturbed, have a measurement smaller than the Information Dimension in the square with 4 sides perturbed, which can be useful in different practical applications.

In Table 3, the same analysis was carried out on the results found for *Box-Counting, compared to the Information Dimension, this time applying* Tsallis entropy to the parameter $q = 0.5$.

Table 3 – Result of the *Box-Counting* Dimension and the Information Dimension with Tsallis entropy

Square	$\boldsymbol{D_{Box}}$	$\boldsymbol{D_{TS}}$
A	1.2796710276704335	0.7899382233835297
B	1.2796710276704335	0.7976331670012815
W	1.2796710276704335	0.8054932828767232
D	1.2796710276704335	0.8126195937450961

Source : The author (2023)

It is observed that the entropy values were accentuated by the parameter q, which can be useful in certain practical problems. This can be exemplified by Figure 31, which represents the variation in Tsallis entropy , in square A.

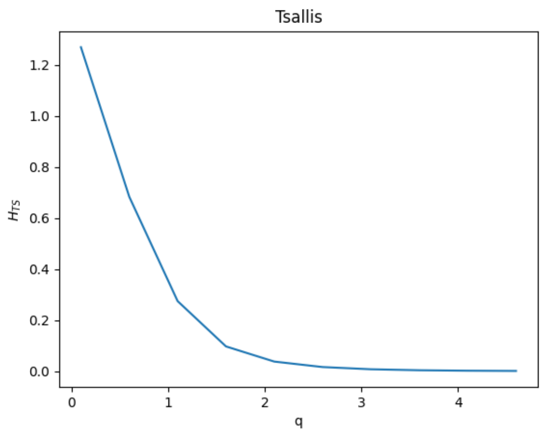
All graphs represent the Information Dimension with Tsallis entropy , which accentuates the entropy value for the entropic parameter q = 0.5 and thus becomes this parameter recommended for tests with other data.

5.2.1 Information Dimension Graphs with Tsallis entropy

Tsallis statistics emerged as an extension of standard statistical mechanics, which is based on Boltzmann-Gibbs-Shannon entropy. Along these lines, the successful applications of Tsallis entropy motivated the researcher to investigate the mathematical tool behind this structure.

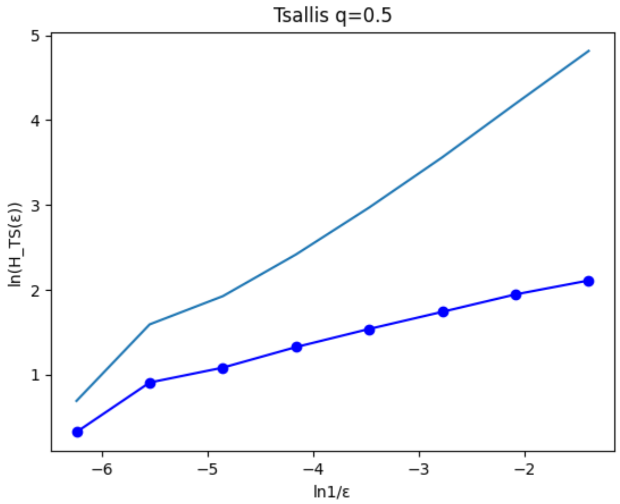
When comparing information dimension analyzes using Shannon entropy and Tsallis entropy , differences in sensitivity to different probability distributions can be observed. Tsallis entropy has been proposed as a generalization of Shannon entropy, which incorporates an additional parameter q. This generalization allows us to capture features of long-range dependence or higher-order dependence that may be present in some complex systems.

Figure 31 – Information Dimension Graph with Tsallis entropy of square A, with 1 side disturbed



Source : The author (2023)

Figure 32 – Information Dimension Graph with Shanonn and Tsallis entropy of square D, with 4 perturbed sides



Source : The author (2023)

Shannon's entropy curve, compared to Tsallis 's in Figure 32, reveals that entropy grows monotonically with the increase in the number of possibilities i, for this entropy the x-axis is the ln (1/ ε) of the height and width of the box square, where the probability distribution is given by the frequency of the figure in each box and the result of the number of *pixels* distributed in each box is the probabilistic uncertainty data linked to the total number of boxes (events). On the y axis is the dimension of information with the ln of entropy.

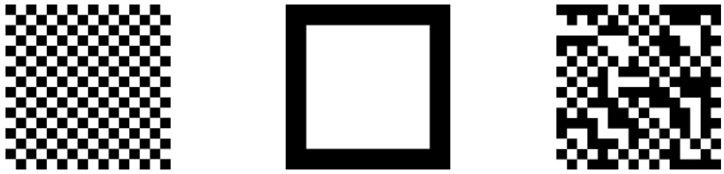
Tsallis entropy are subtle, despite the modified boundaries, it can be seen in Figure 31 that the curve in which the x axis is the q parameter, and this is a measure in the graph, shows a growth between 0 and 1, has higher entropy, showing the entropic index 0.5 of greater relevance to be used in applications in biosystems.

TSallis entropy is mainly a good choice for its implementation.

5.3 EXPECTED RESULTS IN GAP GRAPHICS FOR STANDARDS A, BEC

There are:

Figure 33 – Standards for calculating the lacunarity index



ABC

Source : The author (2023)

Table 4 – Results of the Information dimensions with Shannon and Tsallis entropy in patterns A, B and C

Standard s	D_S	D_{TS}
A	0.3892662822126259	1.0667203264153013
B	0.3957565795875759	1.0505964536385166
W	0.3897991498866205	1.0622756438613044

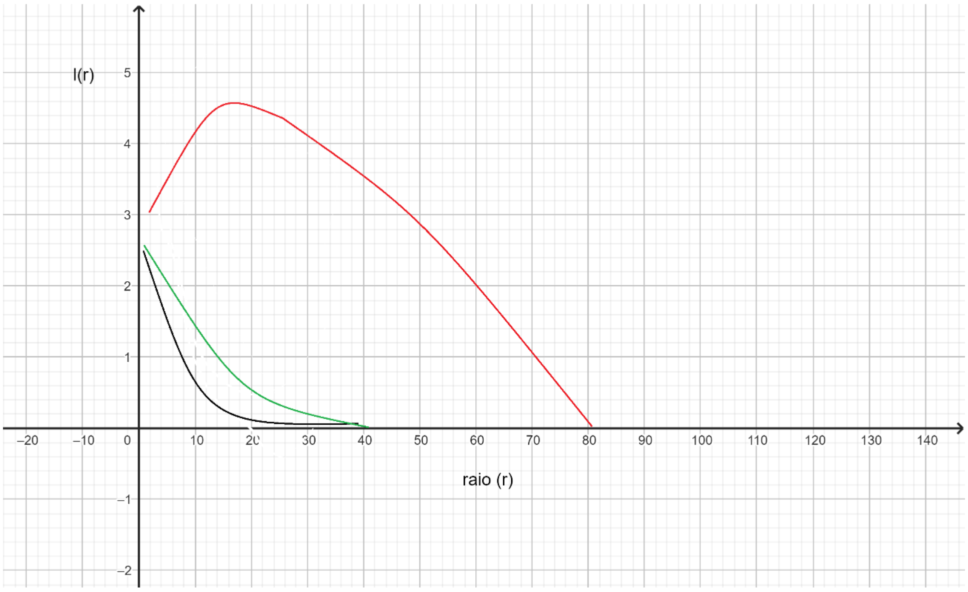
Source: The author (2024)

Patterns A, B and C, of the same mass, when used for the information dimension with Shannon entropy, first measured in table 4, presented results with few changes in the measurement of the dimension and pattern B shows to be the largest measurement in the dimension of information with Shannon entropy. With Tsallis entropy , pattern B presents less information, so its size is smaller, which shows the importance of the presence of the parameter q in identifying changes in the image pattern.

Thus, in pattern B, Shannon's entropy did not signal the texture, while Tsallis ' entropy, because the pattern is simple, guarantees a lower result.

For low values of r, lacunarity is an index only of the percentage or number of pixels in the shape, as well as mass, regardless of its geometry. However, as the radius(r) of the boxes increases, the curve accentuates, and the value of the lacunarity l(r) increases.

Figure 34 – Graph of the expected result of the lacunarity index of patterns A, B and C with radius > 40



ABC

Source : The author (2023)

We observe the lacunarity L = 1 for r =9 in pattern A, L = 1 for r between 10 and 15 in pattern C, however in pattern B, the lacunarity is greater than 3 with the radius still between 0 and 5, the mass distribution, the pixels are more clumped together, the lacunarity value is higher in most of the scale under analysis. These are the expected results , however, we used the sliding box method assigned in the FracDim application, this application is being tested to calculate the lacunarity, which we obtained fast results in radii below 20, the method has already been applied in other studies with radii larger, but there is a need for a machine with a more efficient processor.

Lacunarity has great importance in ecology. For this research, we can proceed with organisms, initially mentioning ants, considering a dimension to characterize the extension of spatial groupings, with each change within the two-dimensional space.

In patterns, the species can be represented in points to obtain the dimension of the validated movement trajectory, based on previous reports.

FINAL CONSIDERATIONS

The present study explored the application of the information dimension, based on Shannon and Tsallis entropy , in detecting boundary perturbations in objects that can be represented by flat figures. It was demonstrated, through specific constructions, that the information dimension is capable of resolving ambiguities generated by the *Box-Counting dimension* , thus obtaining greater precision in the analysis of the complexity of the boundaries of objects modeled in the plane.

The results obtained revealed that squares with subtle disturbances in their contours had the same *Box-Counting dimension* , but different dimensions of information. The sensitivity of entropy to the distribution of information in each box was highlighted, potentially becoming a more effective tool for detecting irregularities.

Furthermore, this work introduced Tsallis entropy , which allowed greater emphasis on the information contained in each box, due to its entropic parameter q. This parameter made it possible to modulate sensitivity to information, which can be useful in different practical applications.

The importance of this study lies in the experimental proof that the information dimension can be successfully applied to distinguish boundary disturbances in flat objects. This approach has the potential to be applied in several areas, including ecology, detection of pathologies in cells and leaves, as well as monitoring areas such as forests.

In summary, the results obtained show the researcher that the information dimension represents a valuable tool when analyzing the detection of boundary disturbances and, consequently, in analyzing the complexity of objects, while the lacunarity index in FracDim corresponded to what was expected, or That is, the pattern with the largest empty space showed greater lacunarity, however, it was also clear that the figure with random patterns can vary the lacunarity depending on some parameters, such as the amount of area of gaps, the dimension of the shape and the radius of the box. . Thus, this study continues to pave the way for new developments and future applications.

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